On a two-loop crossed six-line master integral with two massive lines

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ABSTRACT: We compute the two-loop crossed six-line vertex master integral with two massive lines in dimensional regularisation, and give the result up to the finite part in $D-4$. We apply the differential equation technique, and focus in particular on the purely analytical calculation of the boundary condition which we derive from a three-fold Mellin-Barnes representation. We also describe how the computation of the boundary condition is used to derive three non-trivial relations among harmonic polylogarithms of the sixth root of unity.

Contents

1 Introduction

The computation of higher order perturbative corrections to heavy-to-light currents has recently been an active field of research. The analytical calculation of two-loop QCD corrections to differential semi-leptonic $b \to u$ decays has been carried out simultaneously by several groups [1–4]. However, one of the occurring master integrals, namely the crossed six-line integral with two massive lines, has not been calculated purely analytically by any of the aforementioned groups.

In this article we close this gap by reanalysing the master integral in question. We rederive the result through order $\mathcal{O}((D-4)^0)$ by purely analytical steps, thereby confirming a suggestion in terms of transcendental constants for the boundary condition of the finite part in $D-4$. During the course of the calculation we also found three non-trivial relations among harmonic polylogarithms (HPLs) of the sixth root of unity.

This paper is organised as follows. In section 2 we define the kinematics and our notation and summarize the final result. In section 3 we elaborate on the boundary condition which we derive from a three-fold Mellin-Barnes (MB) representation. In section 4 we show how to derive from the MB expressions three non-trivial relations between HPLs of the sixth root of unity. We conclude in section 5.

2 Definitions and results

In the following we consider the two-loop crossed six-line master integral depicted in figure 1. We work in dimensional regularisation with $D = 4 - 2\epsilon$ dimensions which is used to regularise both UV and IR divergences. Moreover, we define

$$x \equiv \frac{(p_b - p)^2 + i\eta}{m_b^2} \, , \tag{2.1}$$

where $+i\eta$ stems from the $+i\eta$ prescription which we tacitly assume to be included in the propagators. Defining also the symbols

$$\int [dk] \equiv \int \frac{d^D k}{(2\pi)^D}, \qquad S_\Gamma \equiv \frac{1}{(4\pi)^{D/2}\, \Gamma(1-\epsilon)}, \tag{2.2}$$

our starting expression reads

$$I = \int [dk_1] \int [dk_2] \frac{1}{\left[(k_2 + p_b)^2 - m_b^2\right] (k_2 + p)^2 \left[(k_1 + p_b)^2 - m_b^2\right] (k_2 - k_1 + p)^2}$$
$$\times \frac{1}{(k_2 - k_1)^2\, k_1^2}. \tag{2.3}$$

The integral in question was first given in [5] and subsequently used in [1–7]. Also the cases of six massless lines [8–13], as well as one [14], two [15, 16], and four massive lines [17–19] are available in the literature. In the latter case the topology has two master integrals. Whereas the purely massless integral reveals a closed form in terms of hypergeometric functions of unit argument, the cases with massive lines are solved order by order in ϵ by means of the differential equation technique [20–26].

The x-dependence of the integral I is obtained by taking the derivative of I w.r.t. x and plugging in the result of a Laporta reduction [27–29]. This yields a linear combination of the integral I itself and 16 other (fewer-line) master integrals, the latter are known to sufficiently high orders in ϵ [5]. The solution of the differential equation yields the x-dependence except for the boundary condition at $x = 0$.

Contrary to the case of four massive lines [17], the boundary condition at $x = 0$ of the present integral cannot be inferred from the condition that the integral be regular at $x = 0$, but has to be calculated explicitly by means of other techniques. We summarize here our analytical result and postpone the derivation of the boundary condition at $x = 0$ to section 3. Through order $\mathcal{O}((D-4)^0)$ the result is

$$I = -S_\Gamma^2 \,(m_b^2)^{-2-2\epsilon} \left\{ \frac{c^{(-4)}}{\epsilon^4} + \frac{c^{(-3)}}{\epsilon^3} + \frac{c^{(-2)}}{\epsilon^2} + \frac{c^{(-1)}}{\epsilon} + c^{(0)} + \mathcal{O}(\epsilon) \right\} \tag{2.4}$$

with

$$c^{(-4)} = \frac{1}{12\,(1-x)^2},$$
$$c^{(-3)} = -\frac{\ln(1-x)}{3\,(1-x)^2},$$
$$c^{(-2)} = \frac{1}{72\,(1-x)^2} \left[48 \ln^2(1-x) - 5\pi^2 \right],$$
$$c^{(-1)} = \frac{1}{36\,(1-x)^2} \left[-32 \ln^3(1-x) + 10\pi^2 \ln(1-x) - 267\,\zeta_3 \right],$$
$$c^{(0)} = \frac{1}{(1-x)^2} \left[\frac{8}{9} \ln^4(1-x) - \frac{5}{9}\pi^2 \ln^2(1-x) + 8 \ln(1-x)\,\mathrm{Li}_3(x) \right.$$
$$\left. + \frac{65}{3} \ln(1-x)\,\zeta_3 + 4\,\mathrm{Li}_2^2(x) - \frac{167\pi^4}{270} \right]. \tag{2.5}$$
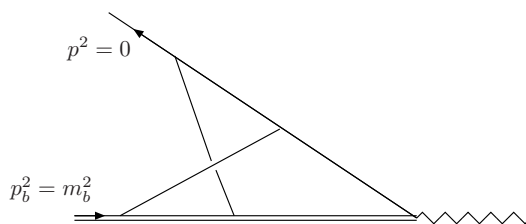
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Figure 1. The two-loop crossed six-line master integral with two massive (double) lines of mass m_b, and four massless (single) lines. The horizontal line has incoming momentum p_b with $p_b^2 = m_b^2$, and the sloped line has outgoing momentum p with $p^2 = 0$.

The value of the coefficient function $c^{(0)}$ at $x = 0$ is one of the main new results of the present article. It confirms a conjecture given in [1, 3] which relies on the PSLQ [30] algorithm. However, we emphasize that all results in the present article were derived solely by analytical steps, and *no fitting* by means of PSLQ or other methods was involved.

3 Computation of the boundary condition

We now turn our attention to the computation of the boundary condition at $x = 0$. Although the coefficient functions $c^{(i)}$ are regular at $x = 0$, this property cannot be used in our case to pin down the integration constant from the differential equation method. Hence the boundary condition has to be computed by other means. Here we choose the Mellin-Barnes technique [31, 32]. We start from the integrand in eq. (2.3), introduce five Feynman parameters to combine the propagators and subsequently integrate over the loop momenta. At this stage we set $x = 0$ and carry out two of the Feynman parameter integrations. After applying analytic continuation formulas for hypergeometric functions [33] we arrive at

$$
\begin{aligned}
I(x = 0) = &-S_\Gamma^2\, \Gamma^2(1-\epsilon)\, \Gamma(1+2\epsilon)\, (m_b^2)^{-2-2\epsilon} \\
&\left\{ -\int_0^1 du\, dy\, dz\, \frac{z^{-1-\epsilon}\, \bar{z}^{-1-\epsilon}\, u^\epsilon\, \bar{u}^{-1-2\epsilon}\, \bar{y}^{-1-2\epsilon}}{(1+2\epsilon)\,(\bar{u}+uy)^{1+2\epsilon}}\, {}_2F_1\left(1+2\epsilon, 1+2\epsilon; 2+2\epsilon; -\frac{y(uy+\bar{u}z)}{z\bar{u}\bar{y}(uy+\bar{u})}\right) \right. \\
&+\int_0^1 du\, dy\, dz\, \frac{z^{-1-\epsilon}\, \bar{z}^{-1-\epsilon}\, u^\epsilon\, \bar{u}^{-1-2\epsilon}\, \bar{y}^{-1-2\epsilon}}{(1+2\epsilon)\,(1+uy)^{1+2\epsilon}}\, {}_2F_1\left(1+2\epsilon, 1+2\epsilon; 2+2\epsilon; -\frac{y(uy+\bar{u}z)}{z\bar{u}\bar{y}(uy+1)}\right) \\
&+\int_0^1 du\, dy\, dz\, \frac{z^{-1+\epsilon}\, \bar{z}^{-1-\epsilon}\, u^{-\epsilon}\, y^{-1-4\epsilon}}{2\epsilon\,\bar{u}\,(1+uy)\,(\bar{z}+uz)^{2\epsilon}}\, {}_2F_1\left(1, 1; 1-2\epsilon; -\frac{uy(\bar{z}+uz)}{z\bar{u}(uy+1)}\right) \\
&\left. -\int_0^1 du\, dy\, dz\, \frac{z^{-1+\epsilon}\, \bar{z}^{-1-\epsilon}\, u^{-\epsilon}\, y^{-1-2\epsilon}}{2\epsilon\,\bar{u}\,(\bar{u}+uy)\,(y+z\bar{u}\bar{y})^{2\epsilon}}\, {}_2F_1\left(1, 1; 1-2\epsilon; -\frac{u(y+z\bar{u}\bar{y})}{z\bar{u}(uy+\bar{u})}\right) \right\},
\end{aligned}
\tag{3.1}
$$

where $\bar{u} = 1 - u$ and similar for y and z. Since each of the hypergeometric functions possesses a one-dimensional Mellin-Barnes (MB) representation according to

$$
{}_2F_1(a, b; c; -z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i}\, \frac{\Gamma(-s)\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)}\, z^s,
\tag{3.2}
$$

we can decompose each of the above terms via a three-dimensional MB representation and subsequently perform the integrations over u, y, and z in terms of Γ-functions. In one of the terms we can apply Barnes' second lemma. This yields

$$I(x=0) = -S_\Gamma^2\,\Gamma^2(1-\epsilon)\,\Gamma(1+2\epsilon)\,(m_b^2)^{-2-2\epsilon}$$

$$\left\{ -\int\limits_{k_1-i\infty}^{k_1+i\infty}\frac{dw_1}{2\pi i}\int\limits_{k_2-i\infty}^{k_2+i\infty}\frac{dw_2}{2\pi i}\int\limits_{k_3-i\infty}^{k_3+i\infty}\frac{dw_3}{2\pi i}\,\Gamma(-\epsilon)\Gamma(-2\epsilon-w_1)\Gamma(2\epsilon+w_1+1)\Gamma(-w_2)\Gamma(2\epsilon+w_1+w_2+1) \right.$$

$$\times\frac{\Gamma(\epsilon+w_1+w_2-w_3+1)\Gamma(2w_1+w_2-w_3+1)\Gamma(-w_3)\Gamma(w_3-w_1)\Gamma(-\epsilon-w_1+w_3)}{\Gamma(2\epsilon+1)\Gamma(-3\epsilon-w_1)\Gamma(2\epsilon+w_1+2)\Gamma(-2\epsilon+w_1+w_2-w_3+1)\Gamma(-2\epsilon-w_1+w_3)}$$

$$\times\Gamma(-4\epsilon-2w_1-w_2+w_3-1)$$

$$+\int\limits_{k_1-i\infty}^{k_1+i\infty}\frac{dw_1}{2\pi i}\int\limits_{k_2-i\infty}^{k_2+i\infty}\frac{dw_2}{2\pi i}\int\limits_{k_3-i\infty}^{k_3+i\infty}\frac{dw_3}{2\pi i}\,\Gamma(-\epsilon)\Gamma(-2\epsilon-w_1)\Gamma(2\epsilon+w_1+1)\Gamma(-w_2)\Gamma(2\epsilon+w_1+w_2+1)$$

$$\times\frac{\Gamma(\epsilon+w_1+w_2-w_3+1)\Gamma(2w_1+w_2-w_3+1)\Gamma(-w_3)\Gamma(w_3-w_1)\Gamma(-\epsilon-w_1+w_3)}{\Gamma(2\epsilon+1)\Gamma(2\epsilon+w_1+2)\Gamma(-\epsilon+w_2+1)\Gamma(-2\epsilon+w_1+w_2-w_3+1)}$$

$$-\int\limits_{k_1-i\infty}^{k_1+i\infty}\frac{dw_1}{2\pi i}\int\limits_{k_2-i\infty}^{k_2+i\infty}\frac{dw_2}{2\pi i}\,\Gamma(-2\epsilon)\Gamma(-\epsilon)\Gamma(\epsilon-w_1)\Gamma(-w_1)\Gamma(w_1+1)\Gamma(-w_2)\Gamma(w_1+w_2+1)$$

$$\times\frac{\Gamma(-4\epsilon+w_1+w_2)\Gamma(-\epsilon+w_1+w_2+1)\Gamma(-4\epsilon+2w_1+w_2+1)}{\Gamma(-2\epsilon+w_1+1)\Gamma(-4\epsilon+w_1+w_2+1)\Gamma(-3\epsilon+w_1+w_2+1)\Gamma(-2\epsilon+w_1+w_2+1)}$$

$$+\int\limits_{k_1-i\infty}^{k_1+i\infty}\frac{dw_1}{2\pi i}\int\limits_{k_2-i\infty}^{k_2+i\infty}\frac{dw_2}{2\pi i}\int\limits_{k_3-i\infty}^{k_3+i\infty}\frac{dw_3}{2\pi i}\,\Gamma(-2\epsilon)\Gamma(-\epsilon)\Gamma(w_1+1)\Gamma(-w_2)\Gamma(w_1+w_2+1)\Gamma(\epsilon-w_1+w_3)$$

$$\times\frac{\Gamma(-\epsilon+w_1+w_2+1)\Gamma(-4\epsilon+w_1+w_2-w_3)\Gamma(-w_3)\Gamma(w_3+1)\Gamma(2\epsilon-w_1+w_3)}{\Gamma(2\epsilon-w_1)\Gamma(-2\epsilon+w_1+1)\Gamma(-4\epsilon+w_1+w_2+1)\Gamma(w_3-w_1)\Gamma(-\epsilon-w_1+w_3)}$$

$$\left.\times\,\Gamma(-2w_1-w_2+w_3-1)\Gamma(-w_1)\right\}. \tag{3.3}$$

We preserve the order of the terms throughout this section. In the above equation (3.3) the contour integrals in the complex plane can be chosen along straight lines parallel to the imaginary axis, i.e. the real parts k_i along the curves are constant. According to refs. [32, 34, 35], these real parts, together with the parameter ϵ, must be chosen in such a way as to have positive arguments in all occurring Γ-functions in order to separate left poles of Γ-functions from right ones. Therefore the four terms in eq. (3.3) are regulated, respectively, for [36]

$$\epsilon = -\frac{5}{128}\,, \qquad k_1 = -\frac{5}{8}\,, \qquad k_2 = -\frac{1}{4}\,, \qquad k_3 = -\frac{9}{16}\,, \tag{3.4}$$

$$\epsilon = -\frac{7}{64}\,, \qquad k_1 = -\frac{9}{32}\,, \qquad k_2 = -\frac{1}{4}\,, \qquad k_3 = -\frac{3}{16}\,, \tag{3.5}$$

$$\epsilon = -\frac{17}{80}\,, \qquad k_1 = -\frac{2}{5}\,, \qquad k_2 = -\frac{3}{40}\,, \tag{3.6}$$

$$\epsilon = -\frac{185}{768}\,, \qquad k_1 = -\frac{229}{384}\,, \qquad k_2 = -\frac{25}{128}\,, \qquad k_3 = -\frac{11}{192}\,. \tag{3.7}$$

We perform the analytic continuation to $\epsilon = 0$ with the package MB [36], which is also used for numerical cross-checks. We then apply Barnes' lemmas and the theorem of residues on the multiple Mellin-Barnes integrals, and insert integral representations of hypergeometric functions as well as

ψ-functions and Euler's B-function where appropriate. The final result reads

$$
\begin{aligned}
I(x=0) = {} & -S_\Gamma^2 \, (m_b^2)^{-2-2\epsilon} \\
& \left\{ \left(-\frac{\pi^2}{24\epsilon^2} + \frac{13\zeta_3}{4\epsilon} - \frac{31\pi^4}{360} \right) \right. \\
& + \left(-\frac{13\zeta_3}{4\epsilon} - \frac{\pi^4}{5} + 24 \, \mathrm{Li}_4\left(\frac{1}{2}\right) + \ln^4 2 - \pi^2 \ln^2 2 + 21\zeta_3 \ln 2 \right) \\
& + \left(-\frac{1}{24\epsilon^4} - \frac{\pi^2}{8\epsilon^2} - \frac{13\zeta_3}{4\epsilon} - \frac{19\pi^4}{360} - 24 \, \mathrm{Li}_4\left(\frac{1}{2}\right) - \ln^4 2 + \pi^2 \ln^2 2 - 21\zeta_3 \ln 2 \right) \\
& \left. + \left(\frac{1}{8\epsilon^4} + \frac{7\pi^2}{72\epsilon^2} - \frac{25\zeta_3}{6\epsilon} - \frac{151\pi^4}{540} \right) + \mathcal{O}(\epsilon) \right\} \\
= {} & -S_\Gamma^2 \, (m_b^2)^{-2-2\epsilon} \left\{ \frac{1}{12\epsilon^4} - \frac{5\pi^2}{72\epsilon^2} - \frac{89\zeta_3}{12\epsilon} - \frac{167\pi^4}{270} + \mathcal{O}(\epsilon) \right\} .
\end{aligned} \tag{3.8}
$$

The calculation of all but the second term in the above equation is more or less straightforward. The second term, however, requires more effort. In this term, at the stage when all integrations have been carried out, we were still left with terms that contain HPLs of the sixth root of unity. Only after the application of the relations given in section 4 we were able to write the result in the form (3.8).

4 Relations between HPLs of the sixth root of unity

Higher transcendental functions that have as arguments powers of the sixth root of unity have been investigated at several places in the literature [37–42]. We introduce the notation

$$
\lambda_\pm = \frac{1}{2} \pm \frac{i}{2}\sqrt{3} = e^{\pm i \frac{\pi}{3}} \tag{4.1}
$$

from which follows immediately $\lambda_+ + \lambda_- = \lambda_+ \lambda_- = 1$. The relations among harmonic polylogarithms of weight four that we found during the course of the calculation of the second term in eq. (3.3) read

$$
\begin{aligned}
& -5 \, \mathrm{HPL}(\{-3,1\}, \lambda_+) - 23 \, \mathrm{HPL}(\{3,-1\}, \lambda_+) - 12 \, \mathrm{HPL}(\{2,-1,-1\}, \lambda_+) \\
& \qquad + 6 \, \mathrm{HPL}(\{2,-1,1\}, \lambda_+) + 6 \, \mathrm{HPL}(\{2,1,-1\}, \lambda_+) \\
& \qquad - 6 \, \mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{1}{4} \ln^4 2 + \pi^2 \ln^2 2 - \frac{1493 \, \pi^4}{38880} + c.c. = 0,
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
& 10 \, \mathrm{HPL}(\{-3,1\}, \lambda_+) + 73 \, \mathrm{HPL}(\{3,-1\}, \lambda_+) - 18 \, \mathrm{HPL}(\{-2,-1,1\}, \lambda_+) \\
& -18 \, \mathrm{HPL}(\{-2,1,-1\}, \lambda_+) + 24 \, \mathrm{HPL}(\{2,-1,-1\}, \lambda_+) + 21 \, \mathrm{HPL}(\{-2,1,1\}, \lambda_+) \\
& \qquad + 21 \, \mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{7}{8} \ln^4 2 - \frac{7}{2} \pi^2 \ln^2 2 + \frac{5083 \, \pi^4}{38880} + c.c. = 0,
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
& -5 \, \mathrm{HPL}(\{-3,1\}, \lambda_+) + 9 \, \mathrm{HPL}(\{3,-1\}, \lambda_+) - 6 \, \mathrm{HPL}(\{-2,1,-1\}, \lambda_+) \\
& \qquad + 3 \, \mathrm{HPL}(\{-2,1,1\}, \lambda_+) + 4 \, \mathrm{HPL}(\{2,1,-1\}, \lambda_+) \\
& \qquad + \mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{1}{24} \ln^4 2 - \frac{5}{6} \pi^2 \ln^2 2 + \frac{791 \, \pi^4}{19440} + c.c. = 0.
\end{aligned} \tag{4.4}
$$

Relations similar to the above ones were investigated systematically in [37]. We find it nevertheless instructive to give a few comments on their derivation.

The first relation, eq. (4.2), can be derived quite easily with the package HPL [43, 44] by applying argument transformations to harmonic polylogarithms and exploiting the fact that $\lambda_+ = 1/\lambda_-$ and $\lambda_+ = 1 - \lambda_-$.

The second one we obtained in the following way. In the second term in eq. (3.3), after the analytic continuation to $\epsilon = 0$, we cannot proceed until the end by merely applying Barnes' Lemmas and corollaries thereof. One way of performing the calculation in this case is to introduce new Feynman parameters as integral representations of higher transcendental functions (Euler's B-function, ψ-functions, hypergeometric functions) in order to be able to perform all MB integrations. Subsequently, one has to integrate over the newly introduced parameters. In our case, there are three such Feynman parameters to be introduced. Note that the three newly introduced Feynman parameters are different and independent from the three-fold integration in (3.1) which lead towards (3.3).[1] At the stage where only one of these new Feynman parameters, say t, is left, we computed the occurring integrand twice by making two different choices of integrating by parts certain terms. The two results are then equated.

The third relation, alike the second, also comes from the second term in (3.3), and also from the stage in where only one new Feynman parameter t is left in the integrand. Here we consider the four integrals

$$
J_1 \equiv \int_0^1 dt \, \frac{\ln(1-t)\ln^2(1-t+t^2)}{t} \, ,
$$

$$
J_2 \equiv \int_0^1 dt \, \frac{\text{Li}_2\big(t-t^2\big)\ln(1-t+t^2)}{t} \, ,
$$

$$
J_3 \equiv \int_0^1 dt \, \frac{\text{Li}_2\bigg(\dfrac{t}{1-t+t^2}\bigg)\ln(1-t+t^2)}{t} \, ,
$$

$$
J_4 \equiv \int_0^1 dt \, \frac{\text{Li}_2\bigg(\dfrac{t^2}{1-t+t^2}\bigg)\ln(1-t+t^2)}{t} \, . \tag{4.5}
$$

We compute each of them separately and subsequently equate them according to

$$
\text{Li}_2\bigg(\frac{t^2}{1-t+t^2}\bigg) + \ln(1-t)\ln(1-t+t^2) + \text{Li}_2\big(t-t^2\big) - \text{Li}_2\bigg(\frac{t}{1-t+t^2}\bigg) = 0 \, , \quad 0 < t < 1 \, . \tag{4.6}
$$

Let us briefly outline the computation of one of the above integrals, namely J_3. We first observe that $\ln(1-t+t^2) = \ln(1-\lambda_+ t) + \ln(1-\lambda_- t)$. Therefore, we only need to compute the integral

$$
\int_0^1 dt \, \frac{\text{Li}_2\bigg(\dfrac{t}{1-t+t^2}\bigg)\ln(1-\lambda_+ t)}{t} \, , \tag{4.7}
$$

which we consider as a function of the complex variable λ_+. We first integrate by parts in order to be able to differentiate the polylogarithm. We then write $1 - t + t^2 = (1-\lambda_+ t)(1-t/\lambda_+)$. In the terms containing $\ln(1-t/\lambda_+)$ we subdivide the interval from $0 \ldots \lambda_+$ and $\lambda_+ \ldots 1$ due to the branch cut of the logarithm. We then take the derivative w.r.t. λ_+, which allows us to perform the integration over t. We finally integrate the result w.r.t. λ_+. The determination of the remaining additive constant is trivial. The computation of the other J_i proceeds along the same lines.

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[1]We tried indeed to compute the second term directly from (3.1), without introducing MB integrations. We succeeded to compute the simple pole, but not the finite part.

Let us conclude this section by giving a few more relations between polylogarithms which turned out to be useful during the calculation. See also [40, 45].

$$\text{Li}_3(1 + \lambda_+) = \frac{13}{18}\,\zeta_3 + \frac{\pi^2}{18}\ln(3) + i\left(\frac{\pi}{10}\,\ln^2(3) + \frac{19\pi^3}{405} - \frac{\Phi(-\frac{1}{3}, 3, \frac{1}{2})}{10\sqrt{3}}\right),$$

(4.8)

$$\text{HPL}(\{-2,2\}, \lambda_+) + \text{HPL}(\{-2,2\}, \lambda_-) = \frac{109\pi^4}{19440} - \frac{\pi^2 \ln^2(3)}{150} - \frac{\Phi(-\frac{1}{3}, 2, \frac{1}{2})^2}{50}$$
$$- \frac{\pi \ln(3)\Phi(-\frac{1}{3}, 2, \frac{1}{2})}{25\sqrt{3}},$$

(4.9)

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{[(k+a)^2]^{s/2}},$$

(4.10)

$$\Phi\left(-\frac{1}{3}, 2, \frac{1}{2}\right) = 4\sqrt{3}\,\text{Im}\left[\text{Li}_2\left(\frac{i}{\sqrt{3}}\right)\right] = -\frac{\pi \ln(3)}{\sqrt{3}} + \frac{10}{\sqrt{3}}\,\text{Cl}_2\left(\frac{\pi}{3}\right),$$

(4.11)

$$\Phi\left(-\frac{1}{3}, 3, \frac{1}{2}\right) = 8\sqrt{3}\,\text{Im}\left[\text{Li}_3\left(\frac{i}{\sqrt{3}}\right)\right].$$

(4.12)

All these relations also allow us to write the integrals J_i in the following simple form

$$J_1 = \frac{271\pi^4}{4860} - \frac{2\pi^2 \ln^2(3)}{75} + \frac{3\,\Phi(-\frac{1}{3}, 2, \frac{1}{2})^2}{25} - \frac{4\pi \ln(3)\Phi(-\frac{1}{3}, 2, \frac{1}{2})}{25\sqrt{3}} - \frac{2\pi\,\Phi(-\frac{1}{3}, 3, \frac{1}{2})}{5\sqrt{3}},$$

(4.13)

$$J_2 = -\frac{139\pi^4}{1944} + \frac{\pi^2 \ln^2(3)}{25} - \frac{2\,\Phi(-\frac{1}{3}, 2, \frac{1}{2})^2}{25} + \frac{6\pi \ln(3)\Phi(-\frac{1}{3}, 2, \frac{1}{2})}{25\sqrt{3}} + \frac{2\pi\,\Phi(-\frac{1}{3}, 3, \frac{1}{2})}{5\sqrt{3}},$$

(4.14)

$$J_3 = -\frac{989\pi^4}{9720} + \frac{\pi^2 \ln^2(3)}{15} + \frac{2\pi \ln(3)\Phi(-\frac{1}{3}, 2, \frac{1}{2})}{5\sqrt{3}} + \frac{2\pi\,\Phi(-\frac{1}{3}, 3, \frac{1}{2})}{5\sqrt{3}},$$

(4.15)

$$J_4 = -\frac{209\pi^4}{2430} + \frac{4\pi^2 \ln^2(3)}{75} - \frac{\Phi(-\frac{1}{3}, 2, \frac{1}{2})^2}{25} + \frac{8\pi \ln(3)\Phi(-\frac{1}{3}, 2, \frac{1}{2})}{25\sqrt{3}} + \frac{2\pi\,\Phi(-\frac{1}{3}, 3, \frac{1}{2})}{5\sqrt{3}}.$$

(4.16)

It is almost needless to say that also in this section all formulas were derived by purely analytical steps.

5 Conclusion

We presented the result of the two-loop crossed vertex master-integral with two massive lines. Our result represents the last missing piece in the fully analytic computation of the two-loop matching coefficients in heavy-to-light decays. The result was obtained by means of the differential equation technique. The boundary condition at $x = 0$ cannot be inferred from the condition that the integral be regular at $x = 0$. Hence we computed it explicitly from a three-dimensional Mellin-Barnes expression.

We emphasize again that all our results have been obtained by purely analytical steps, and *no fitting* of numerical values to transcendental constants by means of PSLQ [30] or other methods has been used.

Acknowledgments

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